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GROUP CLASSIFICATION OF EQUATIONS OF MOTION OF A GAS IN A CONSTANT FORCE FIELD

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A group classification is proposed for a set of equations describing ideal gas flows in a constant force field. This gas motion is denoted by solutions which are invariant with respect to the operator $X_4 + X_{10}$ [1]. The algebraic approach employed is based on the recently developed analysis [1].

1. Factor-System. Solutions of the equation of gas dynamics are considered which are invariant for the $X = X_4 + X_{10} = \partial_t + t\partial_x + \partial_u$ operator from the optimal system [1]. The invariants of this operator are

$$x_1 = x - t^2/2, \quad y, \; z, \; u - t, \; v, \; w, \; \rho, \; p,$$

where t is the time; (x, y, z) are the spatial coordinates; (u, v, w) are the rate; ρ is the density; p is the gas pressure. The invariant solution has the representation

$$u = t + U(x_1, y, z), \quad v = V(x_1, y, z), \quad w = W(x_1, y, z), \quad \rho = \rho(x_1, y, z), \quad p = p(x_1, y, z),$$

and the factor-system

$$d_1\mathbf{v} + \rho^{-1}\nabla_1 p = (-1, 0, 0), \quad d_1\rho + \rho \operatorname{div}_1\mathbf{v} = 0, \quad d_1 p + A(p, \rho)\operatorname{div}_1\mathbf{v} = 0, \tag{1.1}$$

where $\mathbf{v} = (U, V, W)$; $d_1 = U\partial_{x_1} + V\partial_y + W\partial_z$; $\nabla_1 = (\partial_{x_1}, \partial_y, \partial_z)$; $\operatorname{div}_1\mathbf{v} = U_{x_1} + V_y + W_z$.

The above title is used because the factor-system (1.1) coincides with the system (describing steady-state gas flows in a constant field of mass forces) obtained in an axiomatic construction based on assumptions on the character of the gas motion and the external force field. In this case these equations arise as one of the invariant submodels of the conventional equations of gas dynamics. Nonstationary gas flows in a constant field of mass forces were studied in [2, 3]. In [2] one-dimensional nonstationary isoentropic flows were studied. In [3] the method of differential relations was used to construct the simple wave type solutions for the two-dimensional case.

2. Group of Equivalences. The group classification of system (1.1) is performed using a group of equivalences that transforms an arbitrary element A. The operators of equivalence transformations are sought in the form

$$X^e = \xi^{x_1}\partial_{x_1} + \xi^y\partial_y + \xi^z\partial_z + \zeta^U\partial_U + \zeta^V\partial_V + \zeta^W\partial_W + \zeta^\rho\partial_\rho + \zeta^p\partial_p + \zeta^A\partial_A. \tag{2.1}$$

Unlike [4], in this case the operators (2.1) are determined assuming dependence on the arbitrary element A in all coordinates of the infinitesimal operator X^e. Since $A = A(p, \rho)$, operator (2.1) must satisfy the conditions of invariance of the system (1.1) supplemented by the equation

$$A_{x_1} = A_y = A_z = A_U = A_V = A_W = 0. \tag{2.2}$$

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TABLE 1

m	A	p	Extending operators
1	$f(p,\rho)$	$\varphi(\rho,\Im)$	—
2	$pf(p\rho^{-\gamma})$	$\rho^{\gamma}\varphi(\Im\rho)$	$2\gamma Y_2+(\gamma-1)Y_4$
3	$pf(p/\rho)$	$\rho\varphi(\Im\rho)$	Y_2
4	$f(p)$	$\varphi(\Im\rho)$	Y_4
5	$pf(\rho)$	$\Im\varphi(\rho)$	$2Y_2+Y_4$
6	γp	$\Im\rho^{\gamma}$	Y_2, Y_4
7	$f(\rho e^{-p})$	$\ln\rho+\varphi(\Im\rho)$	Y_4-2Y_3
8	$f(\rho)$	$\varphi(\rho)+\Im$	Y_3
9	1	$\ln\rho+\Im$	Y_3, Y_4
10	0	$\Im$	Y_4, Y_λ

TABLE 2

N	$r=1$	Nor
1	$X_7+\alpha X_1$	2,1
2	$X_2+\alpha X_1$	3,1
3	X_1	L_4
	$r=2$	
1	X_7, X_1	2,1
2	$X_2, X_3+\beta X_1$	3,1
3	X_2, X_3	L_4
4	X_2, X_1	3,1
	$r=3$	
1	X_1, X_2, X_3	L_4
2	$X_2, X_3, X_7+\alpha X_1$	L_4

Here, since the functions $\mathbf{v}(x_1,y,z)$, $\rho(x_1,y,z)$, $p(x_1,y,z)$, and $A(\rho,p)$ act in different spaces, the formulas of extension will be different. The coordinates of the extented operator

$$\overline{X}^e = X^e + \zeta^{U_{x_1}}\partial_{U_{x_1}} + \zeta^{U_y}\partial_{U_y} + \zeta^{U_z}\partial_{U_z} + \dots \tag{2.3}$$

can be determined from the formula

$$\zeta^{h_\lambda} = D^e_\lambda\zeta^h - h_{x_1}D^e_\lambda\xi^{x_1} - h_y D^e_\lambda\xi^y - h_z D^e_\lambda\xi^z.$$

Here h takes on the values U, V, W, ρ, p; $\lambda = x_1, y, z$; the operator $D^e_\lambda = \partial_\lambda + U_\lambda\partial_U + V_\lambda\partial_V + W_\lambda\partial_W + \rho_\lambda\partial_\rho + p_\lambda\partial_p + (A_\rho\rho_\lambda + A_p p_\lambda)\partial_A$. The coordinates of the extended operator (2.3) related to the arbitrary element due to (2.2) obey the formula

$$\zeta^{A_\lambda} = \widetilde{D}^e_\lambda\zeta^A - A_\rho\widetilde{D}^e_\lambda\zeta^\rho - A_p\widetilde{D}^e_\lambda\zeta^p \quad (\lambda = x_1, y, z, U, V, W, \rho, p),$$

where $\widetilde{D}^e_\lambda = \partial_\lambda$ $(\lambda = x_1, y, z, U, V, W)$; $\widetilde{D}^e_\rho = \partial_\rho + A_\rho\partial_A$; $\widetilde{D}^e_p = \partial_p + A_p\partial_A$. As in the classical case [4], it can be assumed that the group of equivalence transformations, which is constructed in terms of operator (2.1) and admitted by Eqs. (1.1) and (2.2), transforms system (1.1) preserving its differential structure and changing only its arbitrary element A. The condition of the A-dependence of all coordinates of the infinitesimal operator (2.1) can, in the general case, extend the classical [4] group of equivalences.

For system (1.1) the group of equivalences coincides with the classical one and arises from the operators

$$2(x_1\partial_{x_1}+y\partial_y+z\partial_z)+U\partial_U+V\partial_V+W\partial_W-2\rho\partial_\rho,\ z\partial_y-y\partial_z+W\partial_V-V\partial_W,\ \rho\partial_\rho+p\partial_p+A\partial_A,\ \partial_p,\ \partial_{x_1},\ \partial_y,\ \partial_z.$$

3. Admissible Group. The operator admitted by system (1.1) can be represented in the form

$$X = \xi^{x_1}\partial_{x_1} + \xi^y\partial_y + \xi^z\partial_z + \zeta^U\partial_U + \zeta^V\partial_V + \zeta^W\partial_W + \zeta^\rho\partial_\rho + \zeta^p\partial_p.$$

Calculations show that the integration of the determining equations reduces to the solution of the expressions

$$(k_3-k_1)\rho\frac{\partial A}{\partial\rho} + (k_3p+k_4)\frac{\partial A}{\partial p} = k_3A, \tag{3.1}$$

where the constants k_1, k_2, k_3, and k_4 are related to the coordinates of the infinitesimal operator $\zeta^U = k_1x_1+k_2$, $\zeta^p = k_3p+k_4$. The cores of the main Lie algebras are the operators

$$X_1 = \partial_{x_1},\quad X_2 = \partial_y,\quad X_3 = \partial_z,\quad X_7 = z\partial_y - y\partial_z + W\partial_V - V\partial_W.$$

The X_1 operator is the core center. In this case the enumeration of operators is taken from [1]. The core of

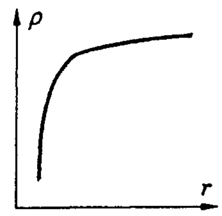

Fig. 1

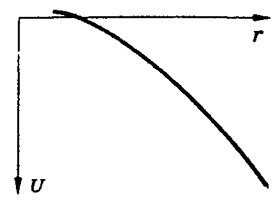

Fig. 2

the main Lie algebras is extended by specializing the $A(p,\rho)$ function. The results of group classification are listed in Table 1, where

$$Y_\lambda=\rho\lambda'(p)\partial_\rho+\lambda(p)\partial_p,\quad Y_2=\rho\partial_\rho+p\partial_p,\quad Y_3=\partial_p,\quad Y_4=2(x_1\partial_{x_1}+y\partial_y+z\partial_z)+U\partial_U+V\partial_V+W\partial_W-2\rho\partial_\rho$$

with the arbitrary $\lambda(p)$ function.

Note that the normalizer factor-algebra of the operator $X = X_4 + X_{10}$ in L_{11} [1] with regard to X $[\mathrm{Nor}_{L_{11}}(X)/X]$ consists of operators $(1,2,3,7)$ (hereafter only the operator numbers are given).

The optimal subalgebra system consists of subalgebras given in Table 2. Hereafter the dimensionality subgroup with number (r,N) is denoted by r and N in Table 2.

We now analyze the invariant solutions of system (1.1).

4. Invariant Solutions. The solutions, invariant with respect to one-dimensional subalgebras, are constructed on the subalgebras $r = 1$ of the optimal system (Table 2).

The invariant solution of subalgebra (1,1) in the cylindrical system of coordinates (x_1,r,θ) has the representation $(x' = x_1 - \alpha\theta)$:

$$U=U(x',r),\quad V=V(x',r),\quad W=W(x',r),\quad \rho=\rho(x',r),\quad p=p(x',r).$$

The factor-system can be written in the vector form

$$B\frac{\partial \mathbf{u}}{\partial x'}+C\frac{\partial \mathbf{u}}{\partial r}=\mathbf{f}, \tag{4.1}$$

where $\mathbf{u}=(u_1,u_2,\dots,u_5)'=(U-\alpha W/r, V, \alpha U/r+W,\rho,p)'$; $\mathbf{f}= -(1,-W^2/r,\ \alpha+u_2u_3;\ \rho V/r, AV/r)'$; E_5 is the unit matrix; $B = u_1E_5 + B^0$; $C = u_2E_5 + C^0$. In the B^0 and C^0 matrices, only $B^0_{15} = (1+\alpha^2/r^2)/\rho$, $B^0_{41}=\rho$, $B^0_{51}=A$, $C^0_{25}=1/\rho$, $C^0_{42}=\rho$, $C^0_{52}=A$ are nonzero. The characteristic equation $\det(B^{-1}C-\lambda E_5)$ of system (4.1) is of the form

$$(\lambda u_1-u_2)^2(\lambda^2(\rho u_1^2-A(1+\alpha^2/r^2))-2\lambda\rho u_1u_2+\rho u_2^2-A)=0.$$

For subalgebra (1,2) (Table 2) the independent variables are $(x_1-\alpha y, z)$, and the factor-system $(x' = x_1-\alpha y)$ is

$$B\frac{\partial \mathbf{u}}{\partial x'}+C\frac{\partial \mathbf{u}}{\partial z}=-\beta^2\mathbf{f}. \tag{4.2}$$

In this case $\beta=(1+\alpha^2)^{-1/2}$; $\mathbf{u}=(u_1,u_2,\dots,u_5)'=(\beta(U-\alpha V),\ \beta(\alpha U+V),\ \beta W,\rho,p)'$; $\mathbf{f}=(1,\alpha,0,0,0)'$; $B=u_1E_5+B^0$ and $C=u_3E_5+C^0$ with nonzero elements in matrices B^0, C^0 ($B^0_{15}=1/\rho$, $B^0_{41}=\rho$, $B^0_{51}=A$, $C^0_{35}=\beta^2/\rho$, $C^0_{43}=\rho$, $C^0_{53}=A$). The characteristic equation of system (4.2) is

$$(\lambda u_1-u_3)^2(\lambda^2(\rho u_1^2-A)-2\lambda\rho u_1u_3+\rho u_3^2-\beta^2A)=0.$$

Finally, for (1,3) the factor-system is of the form

$$B\frac{\partial \mathbf{u}}{\partial y}+C\frac{\partial \mathbf{u}}{\partial z}=-\beta^2\mathbf{f},$$

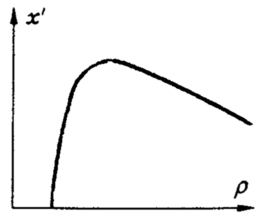

Fig. 3

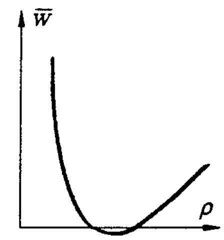

Fig. 4

where $\mathbf{u} = (u_1, u_2, \ldots, u_5)' = (U, V, W, \rho, p)'$; $\mathbf{f} = (-1, 0, 0, 0, 0)'$; $B = u_2 E_5 + B^0$ and $C = u_3 E_5 + C^0$ with nonzero elements in the matrices B^0, C^0 ($B^0_{25} = 1/\rho$, $B^0_{42} = \rho$, $B^0_{52} = A$, $C^0_{35} = 1/\rho$, $C^0_{43} = \rho$, $C^0_{53} = A$). In this case, the characteristics can readily be determined if it is taken into account that for $\mathbf{f} = 0$ and $U = 0$ this system coincides with plane steady-state flows parallel to the plane $x = 0$.

Solutions that are invariant for the two-dimensional subalgebras are constructed on the subalgebras θ_2 of the optimal system (Table 2).

For subalgebra (2,1) the factor-system is a particular case (4.1) with $\alpha = 0$, $\partial \mathbf{u}/\partial x' = 0$. The solution is isoentropic and can be found from a set of two normal differential equations

$$VU' = -1, \quad (V^2 - A/\rho)\rho' = -\rho(V^2 - W^2)/r. \tag{4.3}$$

In this case a prime denotes the r derivative; $V = c_2/(\rho r)$, $W = c_1/r$ with the arbitrary constants c_1, c_2 ($c_2 \neq 0$). The qualitative behavior of the solutions of system (4.3) for a polytropic gas ($\gamma = 1.4$) is shown in Figs. 1 and 2.

The invariant solution (2,2) and (2,3) is a particular solution for system (4.2) with $\alpha = 0$ and independent invariant $x' = x_1 - \beta z$. With $\beta = 0$ [subalgebra (2,3)] for one of the solutions $U = 0$, $p(x')$, $V(x')$, $W(x')$ are arbitrary and $\rho(x') = -p'(x')$. The other solution of this factor-system (for any β) is isoentropic and can be determined from a set of normal differential equations

$$\overline{W}' = -\frac{\beta\rho}{c(1+\beta^2)}, \quad (\rho A - c^2)\rho' = -\frac{\rho^3}{(1+\beta^2)}, \tag{4.4}$$

where $\overline{W} = \sin\theta \cdot U + \cos\theta \cdot W$; $\overline{U} = \cos\theta \cdot U - \sin\theta \cdot W$; $\tan\theta = \beta$; $\overline{U}\rho = c$; $V = 0$. In particular, for a polytropic gas ($A = \gamma p = \gamma c_1 \rho^\gamma$) the general solution of Eqs. (4.4) is

$$\overline{W} = \beta\left(\frac{c}{\rho} + \frac{c_1}{c}\rho^\gamma\right), \quad \frac{c}{2\rho^2} + \frac{\gamma c_1}{(\gamma - 1)}\rho^{\gamma-1} = -\frac{x'}{(1+\beta^2)},$$

from which the invariant solution can be found. Figures 3 and 4 demonstrate the qualitative behavior of the curves $x'(\rho)$ and $\overline{W}(\rho)$.

For the arbitrary function $A(p, \rho)$, the invariant solution (2,4) holds with $\rho = \text{const}$ and $p = \text{const}$. Other solutions can be obtained only for a special type of function $A(p, \rho)$, such as

$$AA_p + \rho A_\rho + A = 0.$$

5. Final Remarks. L. V. Ovsyannikov suggests considering the partially invariant zero type solutions as the "source" of simple solutions (lecture). In gas dynamics for these solutions $p = \text{const}$, $\rho = \text{const}$. The dimensionality of the problem in such solutions can be decreased by integrating along the current lines. In a given submodel:

$$\frac{dU}{ds} = -1, \quad \frac{dV}{ds} = 0, \quad \frac{dW}{ds} = 0, \quad \frac{dx_1}{ds} = U, \quad \frac{dy}{ds} = V, \quad \frac{dz}{ds} = W. \tag{5.1}$$

Let, e.g., for $s = 0$, the initial conditions be given:

$$x_1 = 0, \quad y = y_0, \quad z = z_0, \quad U = U_0(y_0, z_0), \quad V = V_0(y_0, z_0), \quad W = W_0(y_0, z_0). \tag{5.2}$$

Integrating (5.1) and (5.2) the following functions can be determined:

$$U = -s + U_0(y_0, z_0), \quad V = V(y_0, z_0), \quad W = W(y_0, z_0), \quad x = -s^2/2 + sU_0, \quad y = y_0 + sV, \quad z = z_0 + sW.$$

In system (1.1) only the $\mathrm{div}_1 \mathbf{v} = 0$ equation, which in variables (s, y_0, z_0) is quadratic in s, is to be solved. Splitting it in s yields three equations for three unknown functions $[U_0(y_0, z_0), V(y_0, z_0), W(y_0, z_0)]$:

$$\frac{\partial V}{\partial y_0}\frac{\partial W}{\partial z_0} - \frac{\partial V}{\partial z_0}\frac{\partial W}{\partial y_0} = 0; \tag{5.3}$$

$$(VW' - W)\Big[V\frac{\partial U_0}{\partial y_0}\frac{\partial U_0}{\partial z_0} - U_0\frac{\partial U_0}{\partial y_0}\frac{\partial V}{\partial z_0} + W\Big(\frac{\partial U_0}{\partial z_0}\Big)^2 - U_0 W'\frac{\partial U_0}{\partial z_0}\frac{\partial V}{\partial z_0} + \frac{\partial U_0}{\partial z_0}\Big]$$
$$= U_0\Big(\frac{\partial V}{\partial y_0} + W'\frac{\partial V}{\partial z_0}\Big) = 1 + V\frac{\partial U_0}{\partial y_0} + W\frac{\partial U_0}{\partial z_0}. \tag{5.4}$$

In (5.4) $W' = dW/dV$ was used, where $W = W(V)$ is the general solution of (5.3). Thus, the solution of system (1.1) reduces to the integration of nonlinear system (5.4) consisting of two first-order equations with two independent variables. An example of the solution of system (5.4) can be the case with $W = cV$, $c = \text{const}$ (without loss of generality it is assumed that $c = 0$). Hence

$$W = 0, \quad V = V(z_0), \quad U_0 = -y_0/V + \varphi(z_0).$$

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REFERENCES

1. L. V. Ovsyannikov, "Program SUBMODELS, Gas Dynamics," *Prikl. Math. Mekh.*, **58**, No.4, 30–55 (1994).
2. K. P. Stanyukovich, *Nonsteady-State Motions of a Solid Medium* [in Russian], Nauka, Moscow (1971).
3. S. V. Meleshko, "On a class of solutions of a set of quasilinear differential equations with many independent variables," in: *Numerical Methods of Continuum Mechanics* [in Russian], Institute of Theoretical and Applied Mechanics, Sibirian Division, Academy of Sceince of the USSR, **12**, No. 4, (1981), pp. 87–100.
4. L. V. Ovsyannikov, *Group Analysis of Differential Equations* [in Russian] Nauka, Moscow (1978).
5. L. V. Ovsyannikov, "The SUBMODELS program," Preprint. Institute of Hydrodynamics, Sib. Div.. Russian Acad. of Sciences, Novosibirsk (1992).